

# Kernels of L-functions and shifted convolutions

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## 1 Introduction

Let  $f$  be a normalised cuspidal eigenform of even weight  $k$ :

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

for  $\mathrm{SL}_2(\mathbb{Z})$  and let  $L_f(s)$  denote its L-function. The nature of the values  $L_f(n)$  for  $n \geq k$  has long been of interest and is related to Beilinson's conjecture. Thanks to work by Beilinson and Deninger-Scholl it is known that all these values are, up to a power of  $1/\pi$ , periods in the sense of Kontsevich-Zagier (see [10] and the references therein). Since then, Brunault, Rogers-Zudilin and others have, in some cases, established *explicit* expressions of non-critical values as periods and determined other aspects of their algebraic structure.

In [4], C. O'Sullivan and the author established a new characterisation of the field containing an arbitrary value of  $L_f(s)$ , even for  $s$  non-integer. The precise statement is reviewed in the next section (Prop. 2.2) but it can be summarised as stating that ratios of non vanishing L-values of  $f$  belong to the field generated by the Fourier coefficients of  $f$  and of certain double Eisenstein series. These double Eisenstein series were constructed as, in a sense, generalisations to non-integer indices of the usual Rankin-Cohen brackets. To obtain our characterisation we analysed those double Eisenstein series by a method parallel to that of [12]. As in [12], the key was the expression of  $L_f(s)L_f(w)$  ( $s, w \in \mathbb{C}$ ) as a Petersson scalar product of  $f$  against a kernel induced by the double Eisenstein series.

On the other hand, [1] gives a real analytic kernel for a product of L-values (Cor. 11.12 of [1]). In that work, Brown uses Manin's theory of iterated Shimura integrals [7, 8] in order to study multiple modular values. The real analytic kernel is an essential ingredient of Brown's proof that each value  $L_f(n)$  is, (up to a power of  $\pi i$ ) a multiple modular value for  $\mathrm{SL}_2(\mathbb{Z})$ .

Here we show that, in combination with [4], the kernel defined in [1] can be used to give a characterisation of the field of non-critical values that involves recognisable objects. That was an aim we were not able to reach in [4] and the expression in terms of recognisable objects we obtain here is a step in that direction. The recognisable objects are shifted divisor sum Dirichlet series and they have previously been studied in a completely different context.

In general, shifted convolution series are the focus of various lines of investigation because of their natural appearance in moment and other analytic problems. Shifted convolutions

involving divisor sums, in particular, are important for the binary additive divisor problem and other related questions.

The specific shifted convolution appearing here turns out to be essentially the one recently studied by M. Kiral in [5], and which, in its domain of initial convergence is given by

$$D_h(\alpha, \beta; s) := \sum_{\substack{n \in \mathbb{N} \\ n > h}} \frac{\sigma_\alpha(n) \sigma_\beta(n-h)}{n^s}, \quad \alpha, \beta, s \in \mathbb{C}, h \in \mathbb{Z}.$$

Somewhat surprisingly, the obstacle to applying the theory of [4] and [1] towards our sought algebraic information is of analytic nature. The Rankin-Selberg method we need to apply (in (10)) cannot be completed in the form most commonly used. This is because of the non-uniform convergence of a series we must integrate against. We overcome this problem by replacing (in eq. (11)) the holomorphic Poincaré series required for our purposes with a non-holomorphic Poincaré series.

The extra variable introduced in this way allows us to perform the analytic continuation of the shifted convolution obtained at the end of the computation (see (17)). Without this extra variable, the shifted convolution would not converge but now we can use the meromorphic continuation of [5] to obtain a well-defined object. This enables us to prove the main

**Theorem 1.1.** *Let  $f$  be a normalized weight  $k$  cuspidal eigenform for  $SL_2(\mathbb{Z})$  and let  $L_f^*(s)$  denote its completed  $L$ -function. Then, for each integer  $r \geq 1$  we have*

$$\frac{L_f^*(k+r)}{L_f^*(k+1)} \in \mathcal{D}_r \mathcal{D}_1 \mathbb{Q}(f) \text{ if } r \text{ is odd and } \frac{L_f^*(k+r)}{L_f^*(k+2)} \in \mathcal{D}_r \mathcal{D}_2 \mathbb{Q}(f) \text{ if } r \text{ is even.}$$

Here  $\mathcal{D}_m$  is the field generated over  $\mathbb{Q}$  by

$$\pi, i \text{ and } D_l(k+m-2, -m-2; n), (n = k-2, \dots, k-2+m; l \in \mathbb{N}), \text{ if } m \text{ is odd,}$$

by

$$\pi, i \text{ and } D_l(k+m-1, -m-1; n), (n = k-1, \dots, k-1+m; l \in \mathbb{N}), \text{ if } m \text{ is even.}$$

Also,  $\mathbb{Q}(f)$  is the field generated by the Fourier coefficients of  $f$ .

It should be noted that, thanks to the explicit form of the analytic continuation in the region we need it, we even have an expression in terms of an Estermann zeta function.

It would be interesting to see if it is possible to determine the arithmetic nature of values of our shifted divisor sum Dirichlet series so that we obtain information about quotients of  $L$ -values.

## 2 The field of L-values

Let  $f$  be a weight  $k$  cuspidal eigenform

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

for  $\mathrm{SL}_2(\mathbb{Z})$  normalised so that  $a(1) = 1$ . Let  $L_f(s)$  denote its L-function and consider its completed version

$$L_f^*(s) = (2\pi)^{-s} \Gamma(s) L_f(s).$$

We will now define the double Eisenstein series introduced in [4]. Set  $B = \{(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}); n \in \mathbb{Z}\}$ . Also set,

$$c_\gamma := c \quad \text{and} \quad j(\gamma, z) = cz + d \quad \text{for} \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

With the convention  $-\pi < \arg(z) \leq \pi$  define

$$\mathbf{E}_{s,k-s}(z, w) := \sum_{\substack{\gamma, \delta \in B \setminus \Gamma \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} j(\delta, z)^{-k}. \quad (1)$$

As shown in [4], this series can be thought of as a Rankin-Cohen bracket of not necessarily integer index, applied to pair of Eisenstein series. We also define the completed double Eisenstein series

$$\mathbf{E}_{s,k-s}^*(z, w) := \frac{\Gamma(s)\Gamma(k-s)\Gamma(k-w)\zeta(1-w+s)\zeta(1-w+k-s)}{e^{-s\pi/2} 2^{3-w} \pi^{k+1-w} \Gamma(k-1)} \mathbf{E}_{s,k-s}(z, w). \quad (2)$$

In [4] it is proved:

**Theorem 2.1.** *The series  $\mathbf{E}_{s,k-s}^*(z, w)$  converges absolutely and uniformly on compact sets for which  $2 < \mathrm{Re}(s) < k-2$  and  $\mathrm{Re}(w) < \mathrm{Re}(s) - 1, k-1 - \mathrm{Re}(s)$ . It has an analytic continuation to all  $s, w \in \mathbb{C}$  and, as a function of  $z$ , it is a weight  $k$  cusp form for  $\Gamma$ . We have*

$$\langle \mathbf{E}_{s,k-s}^*(\cdot, w), f \rangle = L_f^*(s) L_f^*(w) \quad (3)$$

for any normalised cuspidal eigenform  $f$  of weight  $k$ .

With this theorem, we characterise the field of values of  $L_f^*(s)$  using a method that is motivated by Zagier's technique ([12]). The latter is based on standard (positive integer index) Rankin-Cohen brackets. We will state and prove a slightly more general version of the characterisation given in [4].

For cusp forms  $f_1, f_2, \dots$  we will be denoting by  $\mathbb{Q}(f_1, f_2, \dots)$  the field obtained by adjoining to  $\mathbb{Q}$  the Fourier coefficients of  $f_1, f_2, \dots$ .

**Proposition 2.2.** *Let  $f$  be a normalised cuspidal eigenform  $f$  of weight  $k$  and let  $s_0 \in \mathbb{C}$  such that  $L_f^*(s_0) \neq 0$ . Then, for all  $s, w \in \mathbb{C}$ , with  $L_f^*(s) \neq 0$ ,*

$$\frac{L_f^*(w)}{L_f^*(s)} \in \mathbb{Q}(\mathbf{E}_{s_0,k-s_0}^*(\cdot, w), \mathbf{E}_{s_0,k-s_0}^*(\cdot, s), f).$$

*Proof.* With Th. 2.1 we have

$$\frac{L_f^*(w)}{L_f^*(s)} = \frac{L_f^*(s_0) L_f^*(w)}{L_f^*(s_0) L_f^*(s)} = \frac{\langle \mathbf{E}_{s_0,k-s_0}^*(\cdot, s), f \rangle}{\langle \mathbf{E}_{s_0,k-s_0}^*(\cdot, w), f \rangle} = \frac{\langle \mathbf{E}_{s_0,k-s_0}^*(\cdot, s), f \rangle / \langle f, f \rangle}{\langle \mathbf{E}_{s_0,k-s_0}^*(\cdot, w), f \rangle / \langle f, f \rangle}.$$

By a general result (see, e.g. [9], Lemma 4), the numerator of the last fraction belongs to  $\mathbb{Q}(\mathbf{E}_{s_0,k-s_0}^*(\cdot, s), f)$ . Likewise for the denominator. This implies the result.  $\square$

We state a result which can be deduced from this proposition.

According to Manin's Periods Theorem [6] there are  $\omega^+(f), \omega^-(f) \in \mathbb{R}$  such that

$$L_f^*(s)/\omega^+(f), \quad L_f^*(w)/\omega^-(f) \in \mathbb{Q}(f)$$

for all  $s, w$  with  $1 \leq s, w \leq k-1$  and  $s$  even,  $w$  odd. Although we know that  $\omega^\pm(f)$  are periods (cf. Sect. 3.4 of [10]) and that (when appropriately normalised) their product is  $\langle f, f \rangle$ , little is known about their quotient. However, Prop. 2.2 implies the following characterisation of the field to which their quotient belongs.

**Proposition 2.3.** *Let  $f$  be a normalised cuspidal eigenform of weight  $k$  for  $SL_2(\mathbb{Z})$  such that  $L_f^*(k/2) \neq 0$ . (In particular,  $k \equiv 0 \pmod{4}$ .) Then*

$$\frac{\omega^+(f)}{\omega^-(f)} \in \mathbb{Q}(\pi, i, \mathbf{E}_{k/2, k/2}^*(\cdot, 4), f).$$

*Proof.* Set  $w = 4, s = 3$  and  $s_0 = k/2$  in Prop. 2.2. Since  $L_f^*(4)/\omega^+(f), L_f^*(3)/\omega^-(f) \in \mathbb{Q}(f)$ , we deduce

$$\frac{\omega^+(f)}{\omega^-(f)} \in \mathbb{Q}(\mathbf{E}_{k/2, k/2}^*(\cdot, 3), \mathbf{E}_{k/2, k/2}^*(\cdot, 4), f).$$

With Prop. 2.4 of [4],

$$\mathbf{E}_{k/2, k/2}^*(\cdot, 3) = \mathbf{E}_{(\frac{k}{2}-2)+2, (\frac{k}{2}-2)+2}^*(\cdot, 2+1)$$

equals (up to an factor in  $\mathbb{Q}(\pi, i)$ ) the Rankin-Cohen bracket

$$[E_{k/2}, E_{k/2}]_2 := \sum_{r=0}^2 (-1)^r \binom{k_1+1}{2-r} \binom{k_2+1}{r} E_{k/2}^{(r)} E_{k/2}^{(2-r)}$$

where

$$E_{2m}(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, z)^{-2m} = -\frac{B_{2m}}{4m} + \sum_{n=1}^{\infty} \sigma_{2m-1}(n) e^{2\pi i n z}.$$

From the general theory of Rankin-Cohen brackets,  $[E_{k/2}, E_{k/2}]_2$  is a cusp form, and it is clear that it has rational Fourier coefficients. □

*Remark.* In [11] (5.13), it is shown that, in a different setting (odd weight  $k$  and higher level of the group), this quotient does have a simple explicit expression.

### 3 Brown's kernel

We maintain the notation of the previous section. In [1], F. Brown gives a kernel for a certain product of values of  $L_f(s)$ .

Let  $i, j \geq 0$  be integers and let  $s$  be such that  $i + j + 2 \operatorname{Re}(s) > 2$ . For  $z = x + iy$  set

$$\mathcal{E}_{i,j}^s(z) = \frac{1}{2} \sum_{\gamma \in B \setminus \Gamma} \frac{y^s}{j(\gamma, z)^{i+s} j(\gamma, \bar{z})^{j+s}}.$$

This series converges absolutely in the indicated region and satisfies

$$\mathcal{E}_{i,j}^s(\gamma z) = j(\gamma, z)^i j(\gamma, \bar{z})^j \mathcal{E}_{i,j}^s(z) \quad \text{for all } \gamma \in \Gamma \text{ and } z \in \mathfrak{H}. \quad (4)$$

It further has a meromorphic continuation to the entire complex plane since

$$\mathcal{E}_{i,j}^s(z) = y^{-\frac{i+j}{2}} E_{i-j}(s + \frac{i+j}{2})(z) \quad (5)$$

where  $E_m(s)(z)$  stands for the weight  $m$  Eisenstein series

$$E_m(s)(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s \left( \frac{|j(\gamma, z)|}{j(\gamma, z)} \right)^m.$$

Here  $\Gamma_\infty$  is the stabiliser of  $\infty$ .

For  $m \in \mathbb{Z}_+$  let  $\langle \cdot, \cdot \rangle_m$  be the pairing on real-analytic functions whose product vanishes exponentially at  $\infty$  which is given by the formula of the Petersson scalar product in weight  $m$ :

$$\langle g, h \rangle_m = \int_{\mathfrak{F}} g(z) \overline{h(z)} y^m \frac{dx dy}{y^2}$$

where  $\mathfrak{F}$  is a fundamental domain of  $\text{SL}_2(\mathbb{Z})$ . Notice that we do not require  $g, h$  to be  $\Gamma$ -invariant as in the case of the actual Petersson scalar product.

Corollary 11.12 of [1] implies:

**Theorem 3.1.** *Let  $r \geq 1$  be an integer and let  $a, b \geq 2$  be integers such that*

$$k = 2a + 2b - 2r - 2. \quad (6)$$

*Then for each normalised cuspidal eigenform  $f$  of weight  $k$  and for each  $s \in \mathbb{C}$ , we have*

$$\begin{aligned} & \frac{\pi^{2a-s-k-r}}{2^{1-2a}} \Gamma(s+k+r-2a) \zeta(2s+k+2r-2a) \langle E_{2a} \mathcal{E}_{k+r-2a,r}^s, f \rangle_{k+r} \\ & = L_f^*(s+k+r-1) L_f^*(s+k+r-2a). \end{aligned} \quad (7)$$

Theorem 3.1 combined with Theorem 2.1 implies that  $\mathbf{E}_{s+k+r-1, -s-r+1}^*(z, s+k+r-2a)$  is a holomorphic projection of

$$\frac{\pi^{2a-s-k-r}}{2^{1-2a}} \Gamma(s+k+r-2a) \zeta(2s+k+2r-2a) E_{2a} \mathcal{E}_{k+r-2a,r}^s$$

in the sense that  $\mathbf{E}_{s+k+r-1, -s-r+1}^*(z, s+k+r-2a)$  is a holomorphic cusp form and for all weight  $k$  forms  $f$  we have

$$\begin{aligned} & \frac{\pi^{2a-s-k-r}}{2^{1-2a}} \Gamma(s+k+r-2a) \zeta(2s+k+2r-2a) \langle E_{2a} \mathcal{E}_{k+r-2a,r}^s, f \rangle_{k+r} \\ & = \langle \mathbf{E}_{s+k+r-1, -s-r+1}^*(z, s+k+r-2a), f \rangle_k. \end{aligned} \quad (8)$$

## 4 Fourier coefficients

In this section we will compute the Fourier coefficients of  $\mathbf{E}_{s+k+r-1, -s-r+1}^*(z, s+k+r-2a)$  so that we can apply Prop. 2.2 to deduce Theorem 1.1.

By the formula for Fourier coefficients of a cusp form in terms of inner products against Poincaré series, (8) implies that

$$\text{the } l\text{-th Fourier coefficient of } \mathbf{E}_{s+k+r-1, -s-r+1}^*(z, s+k+r-2a) = \frac{\pi^{2a-s-k-r}(4\pi l)^{k-1}}{2^{1-2a}\Gamma(k-1)} \Gamma(s+k+r-2a) \zeta(2s+k+2r-2a) \langle E_{2a} \mathcal{E}_{k+r-2a, r}^s, P_l \rangle_{k+r} \quad (9)$$

where

$$P_l(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{e^{2\pi i l \gamma z}}{j(\gamma, z)^k}$$

is the  $l$ -th Poincaré series of weight  $k$ .

Therefore, the determination of the field of L-values appearing in Theorem 2.2 reduces to the computation of the inner products

$$\langle E_{2a} \mathcal{E}_{k+r-2a, r}^s, P_l \rangle_{k+r}. \quad (10)$$

We will compute this integral using the Rankin-Selberg method. However, we have to modify slightly the method because otherwise, along the way we obtain a series which cannot be interchanged with integration. To overcome this difficulty we, essentially, regularise the integral. Let

$$P_l^t(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^{\bar{t}} \frac{e^{2\pi i l \gamma z}}{j(\gamma, z)^k}$$

is the  $l$ -th non-holomorphic Poincaré series of weight  $k$ . It is clear that

$$\langle E_{2a} \mathcal{E}_{k+r-2a, r}^s, P_l \rangle_{k+r} = \langle E_{2a} \mathcal{E}_{k+r-2a, r}^s, P_l^t \rangle_{k+r} |_{t=0}. \quad (11)$$

Let  $\text{Re}(t) \gg 0$ . With (4) we now have

$$\begin{aligned} \langle E_{2a} \mathcal{E}_{k+r-2a, r}^s, P_l^t \rangle_{k+r} &= \int_{\mathfrak{F}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{E_{2a}(\gamma z)}{j(\gamma, z)^{2a}} \frac{\mathcal{E}_{k+r-2a, r}^s(\gamma z)}{j(\gamma, z)^{k+r-2a} j(\gamma, \bar{z})^r} \text{Im}(\gamma z)^t \frac{\overline{e^{2\pi i l \gamma z}}}{j(\gamma, \bar{z})^k} y^{k+r} d\mu z \\ &= \int_{\mathfrak{F}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} E_{2a}(\gamma z) \mathcal{E}_{k+r-2a, r}^s(\gamma z) \text{Im}(\gamma z)^{t+k+r} \overline{e^{2\pi i l \gamma z}} d\mu \gamma z. \end{aligned} \quad (12)$$

Since  $E_{2a} \mathcal{E}_{k+r-2a, r}^s$  has polynomial growth at the cusps, we have uniform convergence when  $\text{Re } t$  is large enough and we can complete the unfolding as usual. Then the last integral becomes

$$\int_0^\infty \int_0^1 E_{2a}(z) \mathcal{E}_{k+r-2a, r}^s(z) y^{t+k+r-2} e^{-2\pi l y} e^{-2\pi i l x} dx dy. \quad (13)$$

To complete the computation we need the Fourier expansion of  $E_m(s)(z)$  as given in Prop. 11.2.16 of [2] (or Th. 3.1 of [3]), for *integer*  $s$  with  $s + r - a + k/2 > 0$  which will be the case that interests us. It gives

$$\begin{aligned} \mathcal{E}_{k+r-2a,r}^s(z) &= y^{a-r-\frac{k}{2}} E_{k-2a}(s+r-a+\frac{k}{2})(z) = \\ &= y^{a-r-\frac{k}{2}} a_0(y) + \sum_{n \geq 1} \left( \frac{\sigma_{2s+k+2r-2a-1}(n)}{n^{s+r-a+\frac{k}{2}}} \left( \sum_{a-\frac{k}{2} \leq j \leq s-1+\frac{k}{2}+r-a} \alpha_j^+(4\pi n y)^{-j} \right) e^{-2\pi n y} y^{a-r-\frac{k}{2}} \right) e^{2\pi i n x} \\ &+ \sum_{n \leq -1} \left( \frac{\sigma_{2s+k+2r-2a-1}(|n|)}{|n|^{s+r-a+\frac{k}{2}}} \left( \sum_{\frac{k}{2}-a \leq j \leq s-1+\frac{k}{2}+r-a} \alpha_j^-(4\pi |n| y)^{-j} \right) e^{2\pi n y} y^{a-r-\frac{k}{2}} \right) e^{2\pi i n x}. \end{aligned} \quad (14)$$

Here

$$\begin{aligned} a_0(y) &= \left| \frac{k}{2} - a \right|! \left( \binom{s+r-a+\frac{k}{2}-1+\left|\frac{k}{2}-a\right|}{\left|\frac{k}{2}-a\right|} \Lambda(2s+2r-2a+k) y^{s+r-a+\frac{k}{2}} + \right. \\ &\quad \left. \binom{\left|\frac{k}{2}-a\right|-s-r+a-\frac{k}{2}}{\left|\frac{k}{2}-a\right|} \Lambda(2-2s-2r+2a-k) y^{1-s-r+a-\frac{k}{2}} \right), \end{aligned}$$

where  $\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$  and

$$\alpha_j^\pm := (-1)^j \left( j + \frac{|k-2a|}{2} \right)! \binom{s+r-a+\frac{k}{2}-1+\frac{|k-2a|}{2}}{j+\frac{|k-2a|}{2}} \left( \frac{\pm \frac{k-2a}{2} - s - r + a - \frac{k}{2}}{j \pm \frac{k-2a}{2}} \right).$$

Here  $\binom{a}{b}$  with  $a < 0$  are defined in accordance to the convention that, if  $j \geq 0$ , then

$$\binom{-a+j-1}{j} = (-1)^j \binom{a}{j}.$$

Therefore the coefficient of  $e^{2\pi i l x}$  ( $l > 0$ ) in the Fourier expansion of  $E_{2a}(z) \mathcal{E}_{k+r-2a,r}^s(z)$  is

$$\sum_{n \leq l} b_n(y) \sigma_{2a-1}(l-n) e^{-2\pi(l-n)y}$$

where, if  $n \neq 0$ ,

$$\begin{aligned} b_n(y) &:= \frac{\sigma_{2s+k+2r-2a-1}(|n|)}{|n|^{s+r-a+\frac{k}{2}}} \left( \sum_{\text{sgn}(n)(a-\frac{k}{2}) \leq j \leq s-1+\frac{k}{2}+r-a} \alpha_j^{\text{sgn}(n)} (4\pi |n| y)^{-j} \right) e^{-2\pi |n| y} y^{a-r-\frac{k}{2}} \quad \text{and} \\ b_0(y) &:= y^{a-r-k/2} a_0(y). \end{aligned}$$

Also, for convenience, we set  $\sigma_{2a-1}(0) := -B_{2a}/(4a)$ .

Therefore, with (13) we have that for  $\text{Re } t \gg 0$ ,

$$\langle E_{2a} \mathcal{E}_{k+r-2a,r}^s, P_l^t \rangle_{k+r} = \int_0^\infty e^{-2\pi l y} \left( \sum_{n \leq l} b_n(y) \sigma_{2a-1}(l-n) e^{-2\pi(l-n)y} \right) y^{t+k+r-2} dy. \quad (15)$$

Since the finitely many terms corresponding to  $b_n(y)$  with  $n \geq 0$  can be directly evaluated at  $t = 0$  to give rational linear combinations of powers of  $\pi$  and  $i$ , we will focus on the infinitely many terms indexed by  $n < 0$ .

$$\begin{aligned}
& \int_0^\infty e^{-2\pi ly} \left( \sum_{n<0} b_n(y) \sigma_{2a-1}(l-n) e^{-2\pi(l-n)y} \right) y^{t+k+r-2} dy \\
&= \sum_{\frac{k}{2}-a \leq j \leq s-1+\frac{k}{2}+r-a} \alpha_j^-(4\pi)^{-j} \sum_{n<0} \frac{\sigma_{2s+k+2r-2a-1}(|n|) \sigma_{2a-1}(l-n)}{|n|^{s+r-a+\frac{k}{2}+j}} \int_0^\infty e^{-4\pi(l-n)y} y^{\frac{k}{2}+a-j-1+t} \frac{dy}{y} \\
&= \sum_{\frac{k}{2}-a \leq j \leq s-1+\frac{k}{2}+r-a} \alpha_j^-(4\pi)^{1-\frac{k}{2}-a-t} \Gamma\left(\frac{k}{2}+a-j-1+t\right) \sum_{n>0} \frac{\sigma_{2s+k+2r-2a-1}(n) \sigma_{2a-1}(l+n)}{n^{s+r-a+\frac{k}{2}+j} (n+l)^{\frac{k}{2}+a-j-1+t}}
\end{aligned} \tag{16}$$

Since  $j \leq s-1+\frac{k}{2}+r-a$  we can expand binomially the term  $((n+l)-l)^{s+\frac{k}{2}+r-a-1-j}$ . This, together with the trivial identity  $\sigma_w(n) = n^w \sigma_{-w}(n)$ , imply that the last sum of (16) becomes

$$\begin{aligned}
\sum_{n>0} \frac{\sigma_{-2s-k-2r+2a+1}(n) n^{s+\frac{k}{2}+r-a-1-j} \sigma_{2a-1}(l+n)}{(n+l)^{\frac{k}{2}+a-j-1+t}} &= \sum_{\mu=0}^{s+\frac{k}{2}+r-a-1-j} \binom{s+\frac{k}{2}+r-a-1-j}{\mu} \times \\
& (-l)^{s+\frac{k}{2}+r-a-1-j-\mu} D_l(2a-1, -2s-k-2r+2a+1; \frac{k}{2}+a-j-1+t-\mu) \tag{17}
\end{aligned}$$

where

$$D_l(\alpha, \beta; w) := \sum_{n>l} \frac{\sigma_\alpha(n) \sigma_\beta(n-l)}{n^w}.$$

In view of (11), we aim to investigate the series in (17) at  $t = 0$ . We are mainly interested in the case that least one of  $s+k+r-1$  and  $s+k+r-2a$  is outside the critical strip. In this case, the series  $D_l$  appearing in (17) are not in the initial region of convergence when  $t = 0$ . However, this convolution series has been studied by M. Kiral in [5] and has given the analytic continuation. Here we do not need the full analytic continuation established in [5] but rather a partial extension. Since [5] has not appeared in print yet, we give here a proof of the part of the analytic continuation we need for our purposes.

For  $r$  odd, set

$$s = 1, \quad a = \frac{k+r-1}{2}.$$

These values of  $s$  and  $a$  satisfy the conditions  $s+r-a+k/2 > 0$  and  $a, b = (k-2a+2r+2)/2 \geq 2$  that are required for our construction. Further, with these values of  $s, a$ , the parameter  $j$  in (16) ranges between  $(1-r)/2$  and  $(1+r)/2$ . For each of these values of  $j$ , the parameter  $\mu$  in (17) ranges between 0 and  $\frac{1+r}{2} - j$  and thus  $j + \mu$  ranges between  $(1-r)/2$  and  $(1+r)/2$ .

Therefore, we need to show all shifted convolutions

$$D_l(k+r-2, -r-2, \nu+t); \quad (\nu = k-2, \dots, k-2+r)$$

appearing in (17) have an analytic continuation in a neighbourhood of  $t = 0$ .



To achieve that, we first note that Prop. 6 of [5] can be adjusted to say that, in the region of absolute convergence, we have

$$D_l(k+r-2, -r-2; \nu+t) = \zeta(3+r) \sum_{m=1}^{\infty} m^{-3-r} \sum_{\substack{x \bmod m \\ (x,m)=1}} e^{\frac{-2\pi i x l}{m}} E_l(\nu+t, k+r-2; \frac{x}{m}) \quad (18)$$

where  $E_l(s, \alpha; \frac{x}{m})$  the (truncated) Estermann zeta function, defined, for  $\operatorname{Re}(s) \gg 0$ , by

$$E_l(s, \alpha; \frac{x}{m}) := \sum_{n>l} \frac{\sigma_{\alpha}(n) e^{2\pi i \frac{xn}{m}}}{n^s} \quad s, \alpha \in \mathbb{C}, x/m \in \mathbb{Q}.$$

The (full) Estermann zeta function  $E(s, \alpha; \frac{x}{m}) := E_0(s, \alpha; \frac{x}{m})$  has a meromorphic continuation through its expression in terms of Hurwitz zeta functions  $\zeta(s, x)$ . This expression, in our case, can be stated, for  $\operatorname{Re}(t) \gg 0$  as:

$$E(\nu+t, k+r-2; \frac{x}{m}) = m^{k+r-2-2\nu-2t} \sum_{u,v=1}^m e^{2\pi i \frac{xuv}{m}} \zeta(-k-r+2+\nu+t, \frac{u}{m}) \zeta(\nu+t, \frac{v}{m}).$$

It is well-known that  $\zeta(s, x)$  has a meromorphic continuation to  $\mathbb{C}$ , with only a simple pole at  $s = 1$ . Therefore the RHS is analytic in a neighbourhood of  $t = 0$ , since  $\nu \in \{k-2, \dots, k-2+r\}$ , thus giving the analytic continuation the LHS in this neighbourhood. Further, when  $x \in (0, 1)$ ,  $\zeta(\nu+t, x)$  is bounded by a constant independent of  $x$ , since  $\nu + \operatorname{Re}(t) > 1$ . By the functional equation of  $\zeta(s, x)$  (or, in the case  $\nu = k+r-2$ , the Taylor expansion of  $\zeta(s, x+1)$  at  $x = 0$ ), we deduce that, for  $t$  in a small neighbourhood of 0

$$\zeta(-k-r+2+\nu+t, \frac{u}{m}) \ll_{k,r} m^t + 1$$

Therefore,

$$E(\nu+t, k+r-2; \frac{x}{m}) \ll_{k,r} m^{k+r-2\nu-t} + m^{k+r-2\nu-2t}.$$

Applying this bound to the series of the RHS of (18), we see that each term is holomorphic in a neighbourhood of  $t = 0$  and bounded by  $m^{k-2\nu-t-1} + m^{k-2\nu-2t-1}$ . Therefore, the function  $D_l(k+r-2, -r-2, \nu+t)$  is holomorphic at  $t = 0$  for all  $\nu \in \{k-2, \dots, k-2+r\}$ .

For  $r$  even, set

$$s = 1, \quad a = \frac{k+r}{2}.$$

Working exactly in the same way as above, we find that  $D_l(k+r-1, -r-1, \nu+t)$  is holomorphic at  $t = 0$  for all  $\nu \in \{k-1, \dots, k-1+r\}$ .

*Proof of Theorem 1.1:* As mentioned above, the terms corresponding to non-negative  $n$  in the sum in (15) lead to elements of  $\mathbb{Q}(\pi, i)$ .

Let  $r$  be odd. What we just proved, together with (9) then imply that the  $l$ -th Fourier coefficient of  $\mathbf{E}_{k+r,-r}^*(z, 2)$  belongs to the field

$$\mathcal{D}_r = \mathbb{Q}(\pi, i, D_l(k+r-2, -r-2; k-2), \dots, D_l(k+r-2, -r-2; k-2+r))$$

obtained by adjoining to  $\mathbb{Q}(\pi, i)$  the values of the shifted convolution  $D_l(k+r-2, -r-2; n)$  at  $n = k-2, \dots, k-2+r$ .

On the other hand, (3) implies that

$$\mathbf{E}_{k+r,-r}^*(z, 2) = \mathbf{E}_{2,k-2}^*(z, k+r).$$

Therefore, with Proposition 2.2 for  $s_0 = 2$  and odd  $r' \geq 1$

$$\frac{L_f^*(k+r)}{L_f^*(k+r')} \in \mathcal{D}_r \mathcal{D}_{r'} \mathbb{Q}(f)$$

Setting  $r' = 1$ , we deduce the first case of Theorem 1.1.

Let  $r$  be even. In the same way as above we find that the  $l$ -th Fourier coefficient of  $\mathbf{E}_{k+r,-r}^*(z, 1) = \mathbf{E}_{1,k-1}^*(z, k+r)$  belongs to the field

$$\mathbb{Q}(\pi, i, D_l(k+r-1, -r-1; k-1), D_l(k+r-1, -r-1; k), \dots, D_l(k+r-1, -r-1; k-1+r))$$

Applying Proposition 2.2 with  $s_0 = 1$ , we deduce, for even  $r' > 1$ ,

$$\frac{L_f^*(k+r)}{L_f^*(k+r')} \in \mathcal{D}_r \mathcal{D}_{r'} \mathbb{Q}(f)$$

where  $\mathcal{D}_r$  denotes the field obtained by adjoining to  $\mathbb{Q}(\pi, i)$  the values of the shifted convolution  $D_l(k+r-1, -r-1; n)$  at  $n = k-1, \dots, k-1+r$ . Setting  $r' = 2$ , we deduce the second case of Theorem 1.1.  $\square$

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